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A New Variable Dimension Simplicial Algorithm for Computing Economic Equilibria on $S^n \times R_+^{m}$[1]

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Abstract. In this paper, a new variable-dimension simplicial algorithm is developed to compute economic equilibria on the Cartesian product of the n-dimensional unit price simplex S^n and the m-dimensional production activity space R_+^m. The algorithm differs from other algorithms in the number of directions in which the algorithm may leave the starting point. More precisely, the algorithm has $2^{n+m+1}-2$ rays to leave the starting point, whereas the other algorithms have at most $2^m(n+1)$ rays. The path of points generated by the algorithm can be interpreted as a globally and universally convergent price and production adjustment process. The process as well as the convergence condition are much more natural and economically meaningful than the adjustment processes obtained by other simplicial algorithms. Furthermore, we apply the algorithm to economies with linear production, economies with constant returns to scale, and economies with increasing returns to scale.

Key Words. Economic equilibria, stationary point problems, simplicial algorithms, simplicial subdivisions, adjustment processes, piecewise linear approximations.

1. Introduction

Nowadays, the fixed-point method has been proved as a very powerful tool for solving highly nonlinear numerical problems since the pioneering

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work of Scarf (Ref. 1). It has become a very useful method in economic equilibrium problems, noncooperative games, traffic assignment problems, and engineering problems (see, e.g., Refs. 2–4).

Over the last several years, the existence and computation of economic equilibria on the Cartesian product of the n-dimensional unit price simplex S^n and the m-dimensional production activity space R^m_+ has attracted wide attention (see, e.g., Refs. 5–12). In an equilibrium of an economy, every producer chooses a production activity in order to maximize his profit and prices and activity levels are such that, for every commodity, demand is at most equal to supply. In Van den Elzen, Van der Laan, and Talman (Ref. 6), an economic adjustment process has been introduced to find an equilibrium in an economy with linear production technologies. In Van der Laan and Kremers (Ref. 11), this adjustment process was generalized to an economy with constant returns to scale by solving a sequence of linear stationary point problems on S^n. In the latter method, convergence is not guaranteed. In Hofkes (Ref. 9) and also in Yamamoto and Yang (Ref. 13), other economic applications are considered such as when the production exhibits increasing returns to scale. To find an equilibrium in an economy with nonlinear nondecreasing returns to scale production technologies, the $(n+m+1)$-ray algorithm was developed in Ref. 9 and the $2^m(n+1)$-ray algorithm in Ref. 13, which could date back to Van der Laan and Talman (Ref. 14).

In this paper, we propose a new simplicial algorithm to compute economic equilibria on $S^n \times R^m_+$. Depending on the sign pattern of the function value at an arbitrarily chosen starting point in $S^n \times R^m_+$, the algorithm starts to leave the starting point along one out of $2^{n+m+1}-2$ rays. The algorithm works in a specific triangulation of the underlying space which is introduced in Ref. 13 and called the VK'-triangulation. Moreover, a sufficient condition (which we call limited resource condition) for the existence of an equilibrium is introduced. This condition generalizes the well-known no-production-without-input assumption in the sense that, if an activity level becomes very large, then at least one of the commodities is in excess demand. Under this condition, the path of points followed by the algorithm can be interpreted as a globally and universally convergent price and production level adjustment. Barring degeneracy, the algorithm converges for any starting point and the process whose path is followed by the algorithm simultaneously adjusts prices and activity levels as follows. Initially, the process increases relatively the prices of the commodities with positive excess demand and decreases relatively the prices of the commodities with negative excess demand, while it increases relatively the activity levels of the firms with positive profits and decreases relatively the activity levels of the firms with negative profits. In general, the prices of the commodities with positive (negative) excess demand

and the activity levels of the firms making positive (negative) profits are kept relatively maximal (minimal). From an economic viewpoint, this behavior is very close to the classical tatonnement process in which prices adjust according to the law of demand and supply, i.e., prices increase in case of positive excess demand and decrease when excess demand is negative. The adjustment process developed in this paper is therefore much more intuitive and appealing than the adjustment processes obtained from other simplicial algorithms known so far. Moreover, the convergence condition given in this paper is much more natural and also economically meaningful than those stated for other simplicial algorithms. We remark that the algorithm will also converge under these conditions.

The paper is organized as follows. In Section 2, we give the piecewise linear path of the algorithm and formulate a sufficient condition for the existence of an equilibrium. Section 3 describes the underlying subdivision of $S^n \times R^m_+$. Section 4 discusses the pivot steps and the replacement steps of the algorithm. In Section 5, we are concerned with the application of the process to several typical economic equilibrium models.

2. Path of the Algorithm

We define the n-dimensional unit simplex S^n by

$$S^n = \left\{ p \in R^{n+1}_+ \,\middle|\, \sum_{i=1}^{n+1} p_i = 1 \right\},$$

where R^{n+1}_+ is the nonnegative orthant of the $(n+1)$-dimensional Euclidean space R^{n+1}. Let $f: S^n \times R^m_+ \to R^{n+1} \times R^m$ be a continuous function with $f(p, y) = (f_1(p, y), f_2(p, y))$, for $p \in S^n$ and $y \in R^m_+$. The function $f = (f_1, f_2)$ is assumed to satisfy

$$p^t f_1(p, y) + y^t f_2(p, y) = 0 \qquad \text{for all } p \in S^n \text{ and } y \in R^m_+. \tag{1}$$

Definition 2.1. A pair $(p^*, y^*) \in S^n \times R^m_+$ is an equilibrium if $f(p^*, y^*) \leq 0$, i.e.,

(i) $f_1(p^*, y^*) \leq 0$,
(ii) $f_2(p^*, y^*) \leq 0$.

Let us give some explanation on the above function in economic terms. Let there be a finite number of consumers, m production activities or firms, and $n+1$ commodities in the economy. A vector $p \in S^n$ can be interpreted as a price vector being normalized on the unit simplex, and a vector $y \in R^m_+$ is a vector of activity levels of the firms. Then, $f_{1j}(p, y)$ can be regarded as the net excess demand of commodity j, $j \in \{1, \ldots, n+1\}$, at price vector p and activity level vector y. The function f_2 is related to the profit of the firms, e.g., $f_{2i}(p, y)$ is the profit of the firm i, $i \in \{1, \ldots, m\}$, at price vector p and activity level y when firm i has a unit production level. Condition (1) reflects the Walras law, stating that all consumers spend their income. An equilibrium for this economy is a price vector $p^* \in S^n$ and a production activity level vector $y^* \in R^m_+$ such that, at (p^*, y^*), the excess demand of the consumption sector is at most equal to the net supply of the production sector and no production activity makes a positive profit.

It follows easily from Definition 2.1 that, because of condition (1), an equilibrium $(p^*, y^*) \in S^n \times R^m_+$ has the property that

$$f_{1j}(p^*, y^*) = 0, \qquad \text{if } p_j^* > 0, \tag{2a}$$

$$f_{1j}(p^*, y^*) \leq 0, \qquad \text{if } p_j^* = 0, \tag{2b}$$

$$f_{2i}(p^*, y^*) = 0, \qquad \text{if } y_j^* > 0, \tag{2c}$$

$$f_{2i}(p^*, y^*) \leq 0, \qquad \text{if } y_i^* = 0. \tag{2d}$$

As shown in Ref. 13, the problem is equivalent to the stationary point or variational inequality problem on $S^n \times R^m_+$ with respect to f and is also equivalent to the nonlinear complementarity problem on $S^n \times R^m_+$. In what follows, we will introduce a simplicial algorithm to solve the problem. As applications of the algorithm, several typical economic examples will be discussed later. Let $(u, v) \in S^n \times R^m_+$ be an arbitrarily chosen starting point of the algorithm. For simplicity, we assume that u is an interior point of S^n. Let $b = (b_1, \ldots, b_m)'$ be such that $b_i > v_i$ for all i. To find an equilibrium in $S^n \times R^m_+$, we propose to follow a piecewise linear path of points starting at (u, v). The path traced by the algorithm can be interpreted as the approximate path generated by an adjustment process in which prices and activity levels are simultaneously adjusted. The process generates a piecewise smooth path, denoted by P, of points in $S^n \times R^m_+$ such that, for every point (p, y) on the path, it holds that, for all i, j,

$$p_j/u_j = a, \qquad \text{if } f_{1j}(p, y) < 0, \tag{3a}$$

$$a \leq p_j/u_j \leq \max_h (p_h/u_h), \qquad \text{if } f_{1j}(p, y) = 0, \tag{3b}$$

$$p_j/u_j = \max_h (p_h/u_h), \qquad \text{if } f_{1j}(p, y) > 0, \tag{3c}$$

$$y_i = av_i \qquad \text{if } f_{2i}(p, y) < 0, \tag{3d}$$

$$av_i \leq y_i \leq cv_i + (1-c)b_i, \qquad \text{if } f_{2i}(p, y) = 0, \tag{3e}$$

$$y_i = cv_i + (1-c)b_i, \qquad \text{if } f_{2i}(p, y) > 0, \tag{3f}$$

for certain numbers a and c satisfying

$$0 \leq a \leq 1 \tag{3g}$$

$$c = a, \qquad \text{if } f_1(p, y) \not\leq 0, \tag{3h}$$

$$c \leq a, \qquad \text{if } f_1(p, y) \leq 0. \tag{3i}$$

Notice that (u, v) satisfies (3) for a equal to 1 and that the process will terminate as soon as a becomes equal to zero at say (p^*, y^*). In the latter case,

$$f_{1j}(p^*, y^*) \leq 0, \qquad \text{if } p_j^* = 0,$$

$$f_{1j}(p^*, y^*) \geq 0, \qquad \text{if } p_j^* > 0,$$

$$f_{2i}(p^*, y^*) \leq 0, \qquad \text{if } y_i^* = 0,$$

$$f_{2i}(p^*, y^*) \geq 0, \qquad \text{if } y_i^* > 0.$$

From condition (1), it follows immediately that (p^*, y^*) is an equilibrium of the problem. Under certain regularity and nondegeneracy conditions, the set of points in $S^n \times R_+^m$ satisfying (3) consists of piecewise smooth loops and paths. Exactly one of these paths is the path P, having the starting point (u, v) as an endpoint. In order to guarantee that the path P is bounded, we impose a simple and also economically meaningful condition on the function f.

Assumption (LRC). Limited Resource Condition. There exists a positive number T such that, for each $(p, y) \in S^n \times R_+^m$ with $\max_i y_i \geq T$, there is an index j satisfying $f_{1j}(p, y) > 0$.

The condition says in economic terms that, when one or more firms choose a high production level, the supply of at least one commodity cannot meet the consumers excess demand.

Theorem 2.1. Under Assumption (LRC), the path P in $S^n \times R_+^m$ starting at (u, v) is bounded, and its other endpoint is an equilibrium.

Proof. Suppose that the path P is unbounded. Then, without loss of generality, there is a sequence $\{(p^k, y^k)\}_1^\infty$ satisfying (3), with some of the components of y^k going to infinity. Therefore there exists a positive integer M such that, for each $k \geq M$,

$$\max_i y_i^k \geq \max\{T, \max_i b_i\}.$$

Moreover, since (p^k, y^k) satisfies (3), it holds that, for each $k \geq M$,

$$f_{1j}(p^k, y^k) \leq 0, \qquad \text{for all } j.$$

By assumption, we have that, for each (p^k, y^k) with $k \geq M$, there is an index i such that

$$f_{1i}(p^k, y^k) > 0,$$

which contradicts

$$f_{1i}(p^k, y^k) \leq 0.$$

Hence, the path P is bounded and has another endpoint, say (p^*, y^*). Clearly, (p^*, y^*) is an equilibrium. □

We are now ready to present an economic interpretation of the adjustments of prices and activity levels along the path P defined in (3). The adjustment process starts in (u, v). Barring degeneracy, the vector $f(u, v)$ contains no zeros. In the case where all commodities in the market are in excess supply, the process keeps initially all the prices fixed, while the activity levels of the firms with positive profit are increased with the same proportion and the activity levels of firms with negative profit do not change. Otherwise, the process increases initially the prices of commodities with positive excess demand proportionally and decreases the prices of commodities with negative excess demand proportionally, while the process increases the activity levels of the firms making positive profit and decreases the activity levels of the firms making negative profit with the same proportion. In general, the process adjusts simultaneously the prices and activity levels according to the sign pattern of the excess demand and the profit. The price of a commodity is kept relatively maximal (minimal) if the excess demand of the commodity is positive (negative) and the activity level of a firm is kept the same proportion smaller (larger) if its profit is negative (positive).

The path P of points from (u, v) as defined in (3) is followed through making alternating replacement steps in the VK'-triangulation of $S^n \times R^m_+$ as described in the next section and pivot steps in a linear system of equations. To do so, in system (3) we replace the function f by its piecewise linear approximation F with respect to the VK'-triangulation. The function F is linear on each simplex of the subdivision and coincides with f on the vertices of every simplex. Then, the algorithm traces a piecewise linear path, denoted by $\bar{P}$, in $S^n \times R^m_+$ such that, for every point (p, y) on $\bar{P}$, it holds that, for all i, j,

$$p_j/u_j = a, \qquad \text{if } F_{1j}(p, y) < 0, \tag{4a}$$

$$a \leq p_j/u_j \leq \max_h (p_h/u_h), \qquad \text{if } F_{1j}(p, y) = 0, \tag{4b}$$

$$p_j/u_j = \max_h (p_h/u_h), \qquad \text{if } F_{1j}(p, y) > 0, \tag{4c}$$

$$y_i = av_i, \qquad \text{if } F_{2i}(p, y) < 0, \tag{4d}$$

$$av_i \leq y_i \leq cv_i + (1-c)b_i, \qquad \text{if } F_{2i}(p, y) = 0, \tag{4e}$$

$$y_i = cv_i + (1-c)b_i, \qquad \text{if } F_{2i}(p, y) > 0, \tag{4f}$$

for certain numbers a and c satisfying

$$0 \leq a \leq 1, \tag{4g}$$

$$c = a, \qquad \text{if } F_1(p, y) \not\leq 0, \tag{4h}$$

$$c \leq a, \qquad \text{if } F_1(p, y) \leq 0. \tag{4i}$$

The function $F = (F_1, F_2)$ is given by

$$F(p, y) = \sum_{i=1}^{t+1} \lambda_i f(p^i, y^i),$$

where $\lambda_i, \ldots, \lambda_{t+1} \geq 0$ are such that

$$\sum_{i=1}^{t+1} \lambda_i = 1$$

and

$$(p, y) = \sum_{i=1}^{t+1} \lambda_i (p^i, y^i)$$

is a point in some t-simplex $\sigma(w^1, \ldots, w^{t+1})$ of the triangulation with vertices $w^i = (p^i, y^i)$, $i = 1, \ldots, t+1$. In Section 4, we demonstrate the existence of a piecewise linear path of points satisfying (4) from (u, v) to an approximate equilibrium under Assumption (LRC). In the next section, we describe the VK'-triangulation of $S^n \times R_+^m$ which underlies the algorithm.

3. Simplicial Subdivision

Let I_{n+1} and I_m denote the set of integers $\{1, 2, \ldots, n+1\}$ and $\{1, 2, \ldots, m\}$, respectively. The ith unit vector in R^{n+1} is denoted by

$e_1(j), j\in I_{n+1}$, while $e_2(i)$ is the ith unit vector in R^m, $i\in I_m$. A vector $s=(s_1, s_2)\in R^{n+1}\times R^m$ is said to be a sign vector if

$$s_{1j}\in\{-1, 0, +1\}, \qquad \text{for every } j\in I_{n+1},$$
$$s_{2i}\in\{-1, 0, +1\}, \qquad \text{for every } i\in I_m.$$

For each sign vector s, let

$$I^-(s_1)=\{j\in I_{n+1}|s_{1j}=-1\},$$
$$I^0(s_1)=\{j\in I_{n+1}|s_{1j}=0\},$$
$$I^+(s_1)=\{j\in I_{n+1}|s_{1j}=+1\},$$
$$I^-(s_2)=\{i\in I_m|s_{2i}=-1\},$$
$$I^0(s_2)=\{i\in I_m|s_{2i}=0\},$$
$$I^+(s_2)=\{i\in I_m|s_{2i}=+1\}.$$

Furthermore, let

$$S=\{s\in R^{n+1}\times R^m|s \text{ is a sign vector, which contains at least one } -1 \text{ and one } +1, \text{ and in the case } s_1\geq 0, \text{ there is some } i\in I_m \text{ such that } s_{2i}<0 \text{ and } v_i>0\}.$$

Note that, in the case $v_i>0$ for all $i\in I_m$, there are $2^{n+m+1}-2$ sign vectors in S containing no zeros at all. Each sign vector $s\in S$ will induce a t-dimensional subset $A(s)$ of $S^n\times R^m_+$, where

$$t=t_1+t_2+1, \qquad \text{with } t_1=|I^0(s_1)| \text{ and } t_2=|I^0(s_2)|.$$

It is readily seen that t lies between 1 and $n+m$ and is equal to 1 for the sign vectors in S containing no zeros at all. Therefore, when $v_i>0$ for all $i\in I_m$, there are $2^{n+m+1}-2$ one-dimensional sets or rays, along one of which the algorithm leaves the starting point (u, v).

Definition 3.1. For $s\in S$, the set $A(s)$ is defined to be the set of points $(p, y)\in S^n\times R^m_+$ satisfying the following system of inequalities:

$$p_j/u_j=a, \qquad \text{if } s_{1j}=-1, \tag{5a}$$
$$a\leq p_j/u_j\leq \max_h (p_h/u_h), \qquad \text{if } s_{1j}=0, \tag{5b}$$
$$p_j/u_j=\max_h (p_h/u_h), \qquad \text{if } s_{1j}=+1, \tag{5c}$$

$$y_i = av_i, \qquad \text{if } s_{2i} = -1, \tag{5d}$$

$$av_i \leq y_i \leq cv_i + (1-c)b_i, \qquad \text{if } s_{2i} = 0, \tag{5e}$$

$$y_i = cv_i + (1-c)b_i, \qquad \text{if } s_{2i} = +1, \tag{5f}$$

$$c = a, \qquad \text{if } s_1 \not\leq 0, \tag{5g}$$

$$c \leq a, \qquad \text{if } s_1 \leq 0, \tag{5h}$$

$$0 \leq a \leq 1. \tag{5i}$$

The boundary of a t-dimensional at $A(s)$ consists of the $(t-1)$-dimensional sets $A(s')$ with $s' \in S$ differing in only one component of s being zero in s, and in the case $I^-(s_1) \neq I_{n+1}$ also of the intersection of $A(s)$ with the $(t-1)$-dimensional set $S^n(I^-(s_1)) \times R^m(I^-(s_2))$, where

$$S^n(I^-(s_1)) = \{p \in S^n \mid p_j = 0, \text{ if } j \in I^-(s_1)\}$$

$$R^m(I^-(s_2)) = \{y \in R_+^m \mid y_i = 0, \text{ if } i \in I^-(s_2)\}.$$

According to the description in Section 2, the algorithm leaves (u, v) along the ray $A(s^0)$ for which $s^0 = \text{sign}(f(u, v))$, where, as in the sequel, the sign of a vector is taken componentwise. In general a point $(p, y) \in S^n \times R_+^m$ satisfies (4) if and only if for some sign vector $s \in S$, (p, y) lies in $A(s)$ and $s = \text{sign}(F(p, y))$. The triangulation of $S^n \times R_+^m$, with respect to which the piecewise linear approximation F of f is defined, must be such that it triangulates each $A(s)$, $s \in S$. The VK'-triangulation of $S^n \times R_+^m$, introduced in Yamamoto and Yang (Ref. 13), satisfies this property. To describe this simplicial subdivision, let $s \in S$ be given, let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{t_1})$ be a permutation of the t_1 elements of $I^0(s_1)$, and let r be a sign vector containing no zeros and conforming to s_2; i.e., $r_i = s_{2i}$ whenever $s_{2i} \neq 0$. In the case $I^-(s_1) = I_{n+1}$ or $v_i = 0$, it must hold that $r_i = +1$ when $s_{2i} = 0$. Let K be a subset of I_{n+1}. The projection $(p_1(K), p_2(r))$ of (u, v) is defined by

$$p_1(K) = u, \qquad \text{if } K = \varnothing;$$

otherwise,

$$p_{1j}(K) = \begin{cases} 0, & \text{for } j \notin K, \\ u_j / \sum_{h \in K} u_h, & \text{for } j \in K, \end{cases}$$

$$p_{2i}(r) = \begin{cases} 0, & \text{for } i \in I^-(r), \text{ if } s_1 \not\leq 0, \\ v_i, & \text{for } i \in I^-(r), \text{ if } s_1 \leq 0, \\ b_i, & \text{for } i \in I^+(r). \end{cases}$$

Finally, we assume that $I^0(s_2)=\{i_1, i_2, \ldots, i_{t_2}\}$ with the ordering $i_1<i_2<\cdots<i_{t_2}$.

Definition 3.2. Let s, γ, and r be given as above. The subset $A(s, \gamma, r)$ of $A(s)$ is given as follows:

(a) if $s_1 \not\leq 0$,

$$A(s, \gamma, r)=\Big\{(p, y)\in S^n\times R^m_+ \,|\, (p, y)=(u, v)+\sum_{i=0}^{t-1}\alpha^i q^i,$$
$$\text{where } 0\leq \alpha^{t_1}\leq\cdots\leq\alpha^1\leq\alpha^0\leq 1,$$
$$\text{and for } j=1, \ldots, t_2,\ 0\leq\alpha^{t_1+j}\leq\alpha^0\Big\};$$

(b) if $s_1\leq 0$,

$$A(s, \gamma, r)=\Big\{(p, y)\in S^n\times R^m_+ \,|\, (p, y)=(u, v)+\sum_{i=0}^{t-1}\alpha^i q^i,$$
$$\text{where } 0\leq\alpha^{t_1}\leq\cdots\leq\alpha^1\leq\min\{1, \alpha^0\},$$
$$\text{and for } j=1, \ldots, t_2,$$
$$0\leq\alpha^{t_1+j}\leq\alpha^0, \text{ if } r_{i_j}=+1 \text{ and } s_{2i_j}=0,$$
$$0\leq\alpha^{t_1+j}\leq\alpha^1, \text{ if } r_{i_j}=-1 \text{ and } s_{2i_j}=0\Big\},$$

where the $(n+m+1)$-vector q^0 is defined by

$$q^0=(p_1(I^+(s_1)), p_2(r))-(u, v),$$

the $(n+m+1)$-vector q^j is defined by, for $j=1, \ldots, t_1$,

$$q^j=(p_1(I^+(s_1)\cup\{\gamma_1, \ldots, \gamma_j\}), p_2(r))$$
$$-(p_1(I^+(s_1)\cup\{\gamma_1, \ldots, \gamma_{j-1}\}), p_2(r)),$$

and the $(n+m+1)$-vector q^{t_1+j} is defined by, for $j=1, \ldots, t_2$,

$$q^{t_1+j}=\begin{cases}(0, v_{i_j}e_2(i_j)), & \text{if } r_{i_j}=-1 \text{ and } s_{2i_j}=0,\\ (0, (v_{i_j}-b_{i_j})e_2(i_j)), & \text{if } r_{i_j}=+1 \text{ and } s_{2i_j}=0.\end{cases}$$

It can easily be verified that the dimension of $A(s, \gamma, r)$ equals t and that $A(s)$ is the union of $A(s, \gamma, r)$ over all permutations γ of the elements

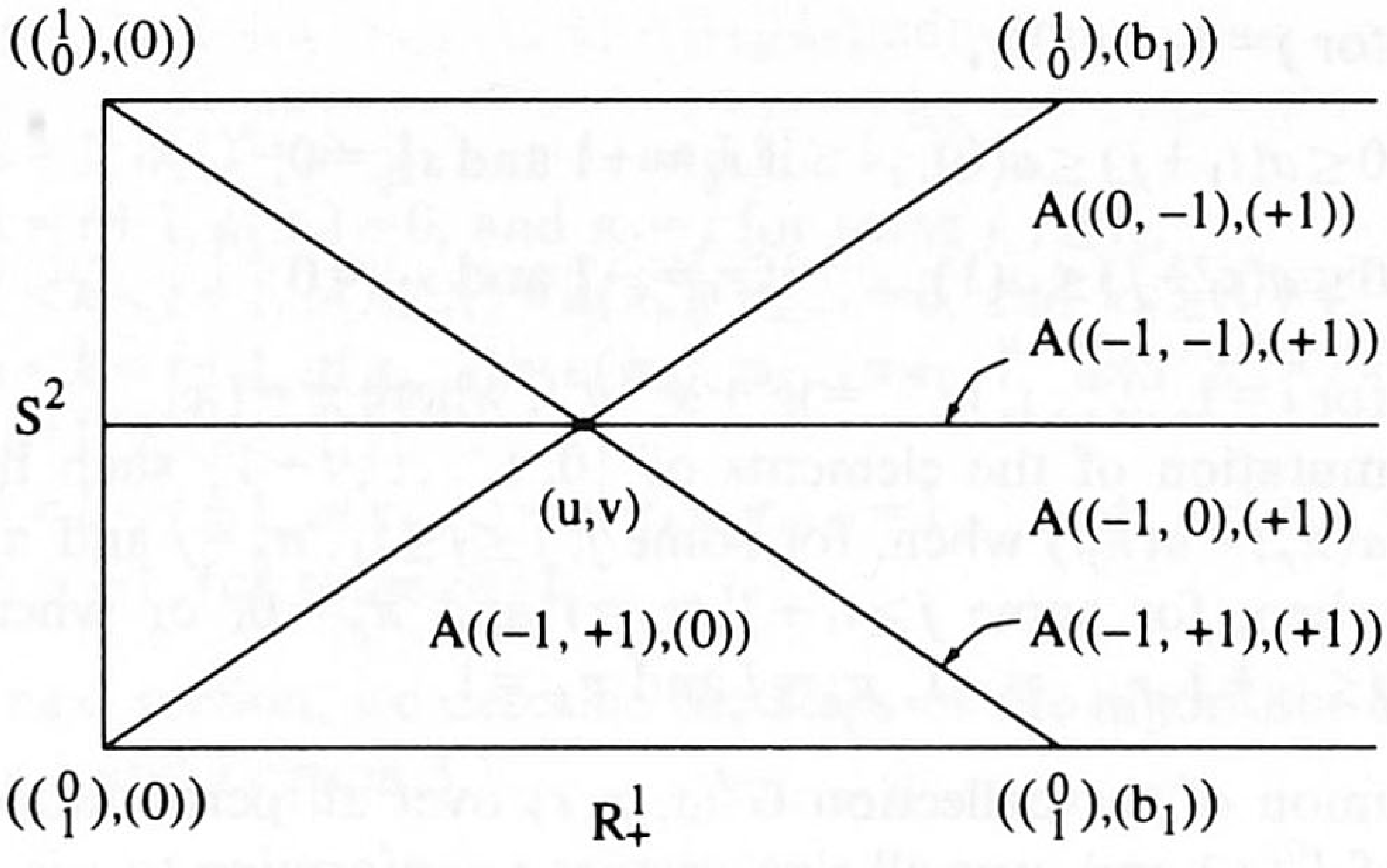

Fig. 1. Subsets $A(s)$ of $S^n \times R^m_+$ for $n=m=1$.

of $I^0(s_1)$ and all sign vectors r conforming to s_2. For $n=m=1$, the subdivision of $S^n \times R^m_+$ is illustrated in Fig. 1.

Let d be a positive integer.

Definition 3.3.

(a) When $s_1 \not\leq 0$, the VK'-triangulation with gridsize d^{-1} of $A(s, \gamma, r)$ is the collection $G^d(s, \gamma, r)$ of t-simplices $\sigma(a, \pi)$ with vertices $w^1, w^2, \ldots, w^{t+1}$ such that:

(i) $w^1 = (u, v) + \sum_{i=0}^{t-1} a(i) d^{-1} q^i$, with $a = (a(0), a(1), \ldots, a(t-1))$ an integer vector such that $0 \leq a(t_1) \leq \cdots \leq a(0) \leq d-1$ and, for $i = t_1 + 1, \ldots, t-1$,

$$0 \leq a(i) \leq a(0);$$

(ii) for $i = 1, \ldots, t$, $w^{i+1} = w^i + d^{-1} q^{\pi_i}$, where $\pi = (\pi_1, \ldots, \pi_t)$ is a permutation of the elements of $\{0, 1, \ldots, t-1\}$ such that $p > p'$ if $a(\pi_p) = a(\pi_{p'})$ when, for some j, $1 \leq j \leq t_1$, $\pi_p = j$ and $\pi_{p'} = j-1$, or when, for some $j \geq t_1 + 1$, $\pi_{p'} = 0$ and $\pi_p = j$.

(b) When $s_1 \leq 0$, the VK'-triangulation with gridsize d^{-1} of $A(s, \gamma, r)$ is the collection $G^d(s, \gamma, r)$ of t-simplices $\sigma(a, \pi)$ with vertices $w^1, w^2, \ldots, w^{t+1}$ such that:

(i) $w^1 = (u, v) + \sum_{i=1}^{t-1} a(i) d^{-1} q^i$, with $a = (a(0), \ldots, a(t-1))$ an integer vector such that $0 \leq a(t_1) \leq \cdots \leq a(1) \leq \min\{a(0), d-1\}$ and, for $j=1, \ldots, t_2$,

$0 \leq a(t_1+j) \leq a(0)$, if $r_{i_j}=+1$ and $s_{2i_j}=0$,

$0 \leq a(t_1+j) \leq a(1)$, if $r_{i_j}=-1$ and $s_{2i_j}=0$;

(ii) for $i=1, \ldots, t$, $w^{i+1}=w^i+d^{-1}q^{\pi_i}$, where $\pi=(\pi_1, \ldots, \pi_t)$ is a permutation of the elements of $\{0, 1, \ldots, t-1\}$ such that $p>p'$ if $a(\pi_p)=a(\pi_{p'})$ when, for some j, $1 \leq j \leq t_1$, $\pi_p=j$ and $\pi_{p'}=j-1$, or when, for some $j \geq t_1+1$, $\pi_p=j$ and $\pi_{p'}=0$, or when for some $j \leq t_1+1$, $r_{i_{j-t_1}}=-1$, $\pi_p=j$ and $\pi_{p'}=1$.

The union of the collection $G^d(s, \gamma, r)$ over all permutations γ of the elements of $I^0(s_1)$ and over all sign vectors r conforming to s is a triangulation of $A(s)$, whereas the union of all these triangulations over all sign vectors $s \in S$ yields the VK'-triangulation of $S^n \times R^m_+$ with gridsize d^{-1}.

Let $\sigma(a, \pi)$ and $\bar{\sigma}(\bar{a}, \bar{\pi})$ be two adjacent simplices in $A(s, \gamma, r)$ with common facet τ opposite to the vertex w^k of σ, $1 \leq k \leq t+1$. Then, $\bar{\sigma}$ is obtained from σ by replacing w^k as described in Table 1, where $e(i)$ is the $(i+1)$th unit vector in R^t, $i=0, 1, \ldots, t-1$.

When a facet lies in the boundary of $A(s, \gamma, r)$, we have the following lemma.

Lemma 3.1.

(a) When $s_1 \not\leq 0$, the facet τ opposite to the vertex w^k of $\sigma(a, \pi)$ in $A(s, \gamma, r)$ lies in the boundary of this set if and only if one of the following cases holds:

(i) $k=1$, $a(0)=d-1$, and $\pi_1=0$;
(ii) $k=t+1$, $a(\pi_t)=0$, and $\pi_t=j$ for some $j, j \geq t_1$;
(iii) $1<k<t+1$, $a(\pi_{k-1})=a(\pi_k)$, $\pi_{k-1}=0$, and $\pi_k \geq t_1+1$;
(iv) $1<k<t+1$, $a(\pi_{k-1})=a(\pi_k)$, $\pi_{k-1}=i-1$, and $\pi_k=i$, for some $i \in \{1, 2, \ldots, t_1\}$.

(b) When $s_1 \leq 0$, the facet τ opposite to the vertex w^k of $\sigma(a, \pi)$ in $A(s, \gamma, r)$ lies in the boundary of this set if and only if one of the following

Table 1. Replacement of the vertex w^k of $\sigma(a, \pi)$.

Value of k	$\bar{\pi}$	$\bar{a}$
$k=1$	$(\pi_2, \ldots, \pi_t, \pi_1)$	$a+e(\pi_1)$
$1<k<t+1$	$(\pi_1, \ldots, \pi_{k-2}, \pi_k, \pi_{k-1}, \ldots, \pi_t)$	a
$k=t+1$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$	$a-e(\pi_t)$

cases holds:

(i) $k=1, a(1)=d-1, \pi_1=1$, and $t_1 \geq 1$;
(ii) $k=t+1, a(\pi_t)=0$, and $\pi_t=j$ for some $j, j\geq t_1$;
(iii) $1<k<t+1, a(\pi_{k-1})=a(\pi_k), \pi_{k-1}=0$, and $\pi_k \geq t_1+1$;
(iv) $1<k<t+1, a(\pi_{k-1})=a(\pi_k), \pi_{k-1}=i-1$, and $\pi_k=i$, for some $i\in\{1, 2, \ldots, t_1\}$;
(v) $1<k<t+1, a(\pi_{k-1})=a(\pi_k), \pi_{k-1}=1$, and $\pi_k=t_1+j$, with $r_{i_j}=-1$, for some $j\in\{1, \ldots, t_2\}$.

In the next section, we describe the steps of the algorithm by making use of Table 1 and Lemma 3.1.

4. Steps of the Algorithm

As stated in Section 2, the algorithm follows a piecewise linear path of points (p, y) in $S^n \times R^m_+$ satisfying (4). The left-hand side of (4) corresponds to the subdivision of $S^n \times R^m_+$ into sets $A(s)$, whereas the right-hand side coincides with the sign pattern of the piecewise linear approximation F of f with respect to the VK'-triangulation. Each point (p, y) on the path $\bar{P}$ lies in $A(s)$ with $s=\operatorname{sign}(F(p, y))$. Let $\sigma(a, \pi)$ with vertices $w^1, \ldots, w^{t+1}$ be a t-simplex in $A(s)$ containing such a point (p, y). Then, there exist unique nonnegative numbers $\lambda_i^*, i=1, \ldots, t+1, \mu_j^*, j\notin I^0(s_1)$, and $\theta_k^*, k\notin I^0(s_2)$, such that

$$\sum_i \lambda_i^* = 1, \qquad (p, y)=\sum_i \lambda_i^* w^i,$$

and

$$\begin{aligned}
F_{1j}(p, y) &= \sum_i \lambda_i^* f_{1j}(w^i) = -\mu_j^*, && \text{if } j\in I^-(s_1),\\
&= \sum_i \lambda_i^* f_{1j}(w^i) = 0, && \text{if } j\in I^0(s_1),\\
&= \sum_i \lambda_i^* f_{1j}(w^i) = \mu_j^*, && \text{if } j\in I^+(s_1),\\
F_{2k}(p, y) &= \sum_i \lambda_i^* f_{2k}(w^i) = -\theta_k^*, && \text{if } k\in I^-(s_2),\\
&= \sum_i \lambda_i^* f_{2k}(w^i) = 0, && \text{if } k\in I^0(s_2),\\
&= \sum_i \lambda_i^* f_{2k}(w^i) = \theta_k^*, && \text{if } k\in I^+(s_2).
\end{aligned}$$

Such a t-simplex is called s-complete. It is readily seen that a t-simplex $\sigma(w^1, \ldots, w^{t+1})$ is s-complete if and only if the $(n+m+2)$-system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(w^i) \\ 1 \end{bmatrix} - \sum_{j \notin I^0(s_1)} \mu_j \begin{bmatrix} s_{1j} e_1(j) \\ 0_2 \\ 0 \end{bmatrix}$$

$$- \sum_{k \notin I^0(s_2)} \theta_k \begin{bmatrix} 0_1 \\ s_{2k} e_2(k) \\ 0 \end{bmatrix} = \begin{bmatrix} 0_1 \\ 0_2 \\ 1 \end{bmatrix} \tag{6}$$

has a nonnegative solution $\lambda_i^*, i=1, \ldots, t+1$, $\mu_j^*, j \notin I^0(s_1)$, and $\theta_k^*, k \notin I^0(s_2)$. The vectors 0_1 and 0_2 in (6) denote the $(n+1)$-vector and the m-vector of zeros, respectively.

Nondegeneracy Assumption. For each solution (λ, μ, θ) of the system (6), at most one of the variables (λ, μ, θ) is equal to zero.

Under this assumption, the set of solutions (λ, μ, θ) of the system (6) forms a line segment, if any. An endpoint of such a line segment is called a basic solution and has exactly one of the variables equal to zero. The line segment of solutions (λ, μ, θ) induces a line segment of points

$$w = \sum_i \lambda_i w^i,$$

in σ, for which, according to (6), it holds that

$$\operatorname{sign}(F(w)) = \operatorname{sign}\left(\sum_i \lambda_i f(w^i) \right) = s.$$

The line segment of solutions (λ, μ, θ) to (6) therefore corresponds to a linear piece of the path $\bar{P}$ in $\sigma(a, \pi)$ and can be followed by making a linear programming pivot step in (6). The algorithm starts with the unique 1-simplex $\sigma^0(w^1, (0))$ in $A(s^0, \varnothing, r^0)$ having $w^1 = (u, v)$ as a vertex, where s^0 is the sign pattern of $f(u, v)$ and $r^0 = s^0$. Clearly, σ^0 is s^0-complete. Notice that, because of the nondegeneracy assumption, s^0 does not contain any zeros. The first piece of the path $\bar{P}$ is contained in σ^0. It can be traced by making a pivot step in (6) with respect to σ^0 by pivoting in the variable λ_2 corresponding to the vertex w^2 of σ^0. After this pivot, either λ_1 becomes zero, μ_j becomes zero for some $j \in \{1, \ldots, n+1\}$, or θ_k becomes zero for some

$k \in \{1, \ldots, m\}$. In general, each linear piece of the path $\bar{P}$ can be followed by making a pivot step in (6) for some simplex $\sigma(a, \pi)$ with vertices $w^1, \ldots, w^{t+1}$ in some $A(s, \gamma, r)$. Suppose that, in such a pivot step μ_j becomes zero for some $j \notin I^0(s_1)$ or θ_k becomes zero for some $k \notin I^0(s_2)$. Then, the corresponding point

$$w^* = (p^*, y^*) = \sum_i \lambda_i w^i$$

is an approximate equilibrium if

$$|I^+(s_1)| + |I^+(s_2)| = 1, \text{ and } s_{1j} = +1 \text{ or } s_{2k} = +1,$$

or if

$$|I^-(s_1)| + |I^-(s_2) \cap \{q | v_q > 0\}| = 1, \text{ and } s_{1j} = -1 \text{ or } s_{2k} = -1.$$

Otherwise, we consider the following four cases.

Case 1. If $\mu_j = 0$ for some $j \in I^-(s_1)$, let $\bar{s}_{1j} = 0$, $\bar{s}_{1h} = s_{1h}$ for $h \neq j$, $\bar{s}_2 = s_2$. Then, σ is a facet of a $(t+1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \bar{\gamma}, r)$, where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{t_1}, j)$, $\bar{a}(l) = a(l)$ for $l = 0, 1, \ldots, t_1$, $\bar{a}(t_1 + 1) = 0$, $\bar{a}(t_1 + l + 1) = a(t_1 + l)$ for $l = 1, \ldots, t_2$, and $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_t, t_1 + 1)$, where

$$\begin{aligned} \bar{\pi}_i &= \pi_i, && \text{if } \pi_i < t_1 + 1, \\ \bar{\pi}_i &= \pi_i + 1 && \text{if } \pi_i > t_1 + 1. \end{aligned}$$

Case 2. If $\mu_j = 0$ for some $j \in I^+(s_1)$, let $\bar{s}_{1j} = 0$, $\bar{s}_{1h} = s_{1h}$ for $h \neq j$, $\bar{s}_2 = s_2$. Then, σ is a facet of a $(t+1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \bar{\gamma}, r)$, where $\bar{\gamma} = (j, \gamma_1, \ldots, \gamma_{t_1})$, $\bar{a}(0) = a(0)$, $\bar{a}(1) = a(0)$, $\bar{a}(l) = a(l-1)$ for $l = 2, \ldots, t$, $\bar{\pi} = (\rho_1, \ldots, \rho_h, 1, \rho_{h+1}, \ldots, \rho_t)$, with $\rho_h = \pi_h$ for $\pi_h = 0$, $\rho_l = \pi_l + 1$ for $l \neq h$.

Case 3. If $\theta_k = 0$ for some $k \in I^-(s_2)$, let $\bar{s}_1 = s_1$, $\bar{s}_{2k} = 0$, and $\bar{s}_{2h} = s_{2h}$ for $h \neq k$. Then, σ is a facet of the $(t+1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \gamma, \bar{r})$ where, if $I^-(s_1) \neq I_{n+1}$ and $v_k \neq 0$, then $\bar{r}_k = -1$, $\bar{r}_k = r_h$ for $h \neq k$, $\bar{a}(l) = a(l)$ for $l = 0, 1, \ldots, t_1 + j$, $\bar{a}(t_1 + j + 1) = 0$, and $\bar{a}(t_1 + l + 1) = a(t_1 + l)$ for $l = j+1, \ldots, t_2$, with j the largest index for which $i_j < k$, and $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_t, t_1 + j + 1)$ with

$$\begin{aligned} \bar{\pi}_h &= \pi_h, && \text{if } \pi_h < t_1 + j + 1, \\ \bar{\pi}_h &= \pi_h + 1, && \text{if } \pi_h > t_1 + j + 1, \end{aligned}$$

and where, if $I^-(s_1) = I_{n+1}$ or $v_k = 0$, then $\bar{r}_k = +1$, $\bar{r}_h = r_h$ for $h \neq k$, $\bar{a}(l) = a(l)$ for $l = 0, 1, \ldots, j$, $\bar{a}(j+1) = a(0)$ and $\bar{a}(l+1) = a(l)$ for $l = j+1, \ldots, t_2$, with j the largest index for which $i_j < k$, and $\bar{\pi} = (\rho_1, \ldots, \rho_h, j+1, \rho_{h+1}, \ldots, \rho_t)$

with $\rho_h=0$ for $\pi_h=0$,

$$\rho_l=\pi_l, \qquad \text{if } \pi_l<j+1, \text{ for } l\neq h,$$
$$\rho_l=\pi_l+1, \qquad \text{if } \pi_l>j+1.$$

Case 4. If $\theta_k=0$ for some $k\in I^+(s_2)$, let $\bar{s}_1=s_1$, $\bar{s}_{2k}=0$, and $\bar{s}_{2h}=s_{2h}$ for $h\neq k$. Then, σ is a facet of the $(t+1)$-simplex $\bar{\sigma}(\bar{a},\bar{\pi})$ in $A(\bar{s},\gamma,r)$ with $\bar{a}(l)=a(l)$ for $l=0,1,\ldots,t_1+j$, $\bar{a}(t_1+j+1)=0$, and $\bar{a}(t_1+l+1)=a(t_1+l)$ for $l=j+1,\ldots,t_2$, with j the largest index for which $i_j<k$, and $\bar{\pi}=(\bar{\pi}_1,\ldots,\bar{\pi}_t,t_1+j+1)$, where

$$\bar{\pi}_h=\pi_h, \qquad \text{if } \pi_h<t_1+j+1,$$
$$\bar{\pi}_h=\pi_h+1, \qquad \text{if } \pi_h>t_1+j+1.$$

In the above four cases, the next linear piece of the path $\bar{P}$ is contained in $\bar{\sigma}$. This linear piece can be followed by making a pivot step in (6) with $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$, where $\bar{w}$ is the vertex of $\bar{\sigma}$ not contained in σ.

If after a pivot step in (6), λ_k becomes zero for some $k\in\{1,\ldots,t+1\}$, then the point

$$w^*=(p^*,y^*)=\sum_{i\neq k}\lambda_i w^i$$

lies in the facet τ of σ opposite to the vertex w^k. The following eight cases may happen according to Lemma 3.1.

Case 1. $k=1$, $a(0)=d-1$, $\pi_1=0$, and $s_1\not\leq 0$. The algorithm terminates with an approximate equilibrium w^*.

Case 2. $t_1\geq 1$, $k=1$, $a(1)=d-1$, and $\pi_1=1$. The algorithm terminates with an approximate equilibrium w^*.

Case 3. $k=t+1$, $a(t_1)=0$, and $\pi_t=t_1$. Then, τ is the $(t-1)$-simplex $\bar{\sigma}(\bar{a},\bar{\pi})$ in $A(\bar{s},\bar{\gamma},r)$ with $\bar{s}_{1j}=-1$ for $j=\gamma_{t_1}$, $\bar{s}_{1h}=s_{1h}$ for $h\neq j$, $\bar{s}_2=s_2$, $\bar{\gamma}=(\gamma_1,\ldots,\gamma_{t_1-1})$, $\bar{a}(l)=a(l)$ for $l=0,1,\ldots,t_1-1$, $\bar{a}(l)=a(l+1)$ for $l=t_1,\ldots,t-2$, and $\bar{\pi}=(\bar{\pi}_1,\ldots,\bar{\pi}_{t-1})$, where

$$\bar{\pi}_h=\pi_h, \qquad \text{if } \pi_h<t_1,$$
$$\bar{\pi}_h=\pi_h-1, \qquad \text{if } \pi_h>t_1.$$

The algorithm continues in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by making a pivot step in (6) with

$$\begin{bmatrix} \bar{s}_{1j} e_1(j) \\ 0_2 \\ 0 \end{bmatrix}.$$

Case 4. $k = t+1$, $a(t_1+j) = 0$ for some j, $1 \le j \le t_2$, and $\pi_t = t_1 + j$. Then, τ is the $(t-1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \gamma, r)$ with $\bar{s}_1 = s_1$, $\bar{s}_{2i_j} = r_{i_j}$, $\bar{s}_{2h} = s_{2h}$ for $h \ne i_j$, $\bar{a}(l) = a(l)$ for $l = 0, 1, \ldots, t_1 + j - 1$, $\bar{a}(l) = a(l+1)$ for $l = t_1 + j - 1, \ldots, t-2$, and $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_{t-1})$, where

$$\bar{\pi}_h = \pi_h, \qquad \text{if } \pi_h < t_1 + j,$$
$$\bar{\pi}_h = \pi_h - 1, \qquad \text{if } \pi_h > t_1 + j.$$

The algorithm proceeds in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by performing a pivot step in (6) with

$$\begin{bmatrix} 0_1 \\ \bar{s}_{2i_j} e_2(i_j) \\ 0 \end{bmatrix}.$$

Case 5. $1 < k < t+1$, $\pi_{k-1} = 0$, $\pi_k = 1$, and $a(0) = a(1)$. Then, τ is the $(t-1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \bar{\gamma}, r)$ with $\bar{s}_{1j} = +1$ for $j = \gamma_1$, $\bar{s}_{1h} = s_{1h}$ for $h \ne j$, $\bar{s}_2 = s_2$, $\bar{\gamma} = (\gamma_2, \ldots, \gamma_{t_1})$, $\bar{a}(0) = a(0)$, $\bar{a}(l) = a(l+1)$ for $l = 1, \ldots, t-2$, and $\bar{\pi} = (\rho_1, \ldots, \rho_{k-2}, \rho_k, \ldots, \rho_t)$, where

$$\rho_h = \pi_h - 1, \qquad \text{for } h = 1, \ldots, k-2,$$
$$\rho_h = \pi_h - 1, \qquad \text{for } h = k, \ldots, t.$$

The algorithm is continued in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by bringing $\begin{bmatrix} \bar{s}_{1j} e_1(j) \\ 0_2 \\ 0 \end{bmatrix}$ into (6).

Case 6.

(i) $s_1 \not\le 0$, $1 < k < t+1$, $\pi_{k-1} = 0$, $\pi_k = t_1 + j$ for some j, $1 \le j \le t_2$, and $a(t_1 + j) = a(0)$. Then, τ is a facet of $\bar{\sigma}(a, \pi)$ in $A(s, \gamma, \bar{r})$ with $\bar{r}_{i_j} = -r_{i_j}$ and $\bar{r}_h = r_h$ for $h \ne i_j$. The algorithm continues in $\bar{\sigma}(a, \pi)$ by making a pivot step in (6) with $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$, where $\bar{w}$ is the vertex of $\bar{\sigma}$ not contained in τ.

(ii) $I^{-}(s_1)=I_{n+1}$, $1<k<t+1$, $\pi_{k-1}=0$, $\pi_k=j$ for some j, $1\leq j\leq t_2$. Then, τ is the $(t-1)$-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(\bar{s}, \gamma, \bar{r})$ with $\bar{s}_1=s_1$, $\bar{s}_{2i_j}=-1$, $\bar{r}_{i_j}=-1$, $\bar{s}_{2h}=s_{2h}$ and $\bar{r}_h=r_h$ for $h\neq i_j$, and $\bar{a}(l)=a(l)$ for $l=0, 1, \ldots, j-1$, $\bar{a}(l)=a(l+1)$ for $l=j, \ldots, t-2$, and $\bar{\pi}=(\rho_1, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_t)$, where

$$\rho_h=\pi_h, \qquad \text{if } \pi_h<j,$$
$$\rho_h=\pi_h-1, \qquad \text{if } \pi_h>j.$$

The algorithm proceeds in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by making a pivot step in (6) with

$$\begin{bmatrix} 0_1 \\ \bar{s}_{2i_j} e_2(i_j) \\ 0 \end{bmatrix}.$$

(iii) $s_1\leq 0$, $|I^0(s_1)|\geq 1$, $1<k<t+1$, $\pi_{k-1}=0$, $\pi_k=t_1+j$ for some j, $1\leq j\leq t_2$, $a(t_1+j)=a(0)$, and $r_{i_j}=+1$. Then, τ is a facet of $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(s, \gamma, \bar{r})$ with $\bar{r}_{i_j}=-1$, $\bar{r}_h=r_h$ for $h\neq i_j$, $\bar{a}(t_1+j)=a(1)$, $\bar{a}(l)=a(l)$ for $l\neq t_1+j$, and $\bar{\pi}$ is the same as π except that π_k moves to behind 1. The algorithm is continued in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by making a pivot step in (6) with $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$, where $\bar{w}$ is the vertex of $\bar{\sigma}$ not contained in τ.

(iv) $s_1\leq 0$, $1<k<t+1$, $\pi_{k-1}=1$, $\pi_k=t_1+j$ for some j, $1\leq j\leq t_2$, $a(1)=a(t_1+j)$, and $r_{i_j}=-1$. Then, τ is a facet of $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $A(s, \gamma, \bar{r})$ with $\bar{r}_{i_j}=+1$, $\bar{r}_h=r_h$ for $h\neq i_j$, $\bar{a}(t_1+j)=a(0)$, $\bar{a}(l)=a(l)$ for $l\neq t_1+j$, $\bar{\pi}$ is the same as π except that π_k moves to behind 0. The algorithm continues in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by pivoting $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$ in (6), where $\bar{w}$ is the vertex of $\bar{\sigma}$ not contained in τ.

Case 7. $1<k<t+1$, $\pi_{k-1}=i-1$, $\pi_k=i$ for some $i\in\{2, \ldots, t_1\}$, and $a(\pi_{k-1})=a(\pi_k)$. Then, τ is a facet of $\bar{\sigma}(a, \pi)$ in $A(s, \bar{\gamma}, r)$ with $\bar{\gamma}=(\gamma_1, \ldots, \gamma_{i-2}, \gamma_i, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{t_1})$. The algorithm proceeds in $\bar{\sigma}(a, \pi)$ by bringing $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$ in (6), where $\bar{w}$ is the vertex of σ not contained in τ.

Case 8. In all other cases, $\bar{\sigma}(\bar{a}, \bar{\pi})$ is adapted according to Table 1 by replacing w^k. The algorithm continues in $\bar{\sigma}(\bar{a}, \bar{\pi})$ by pivoting $\begin{bmatrix} f(\bar{w}) \\ 1 \end{bmatrix}$ in (6), where $\bar{w}$ is the vertex of $\bar{\sigma}$ not contained in τ.

This completes the description of how the algorithm operates on $S^n \times R_+^m$. We are now ready to discuss the convergence of the algorithm. As norm we denote $\|\cdot\|_\infty$ by

$$\|x\|_\infty = \max_i |x_i|, \qquad \text{for } x \in R^n.$$

Lemma 4.1. Let

$$D(T) = \left\{(p, y) \in S^n \times R_+^n \mid \max\left\{T, \max_i b_i\right\} \right.$$

$$\left. \leq \max_i y_i \leq \max_i b_i + \max\left\{T, \max_i b_i\right\}\right\},$$

and let

$$\varphi = \inf\left\{\max_i f_{1i}(p, y) | (p, y) \in D(T)\right\}.$$

If Assumption (LRC) is satisfied, then $\varphi > 0$.

Proof. The conclusion directly follows from the compactness of $D(T)$ and the continuity of f. □

Due to the compactness of $D(T)$, the function f is uniformly continuous on $D(T)$. Therefore, for a positive $\epsilon \leq \varphi/2$, there is a $\delta > 0$ such that

$$x, y \in D(T) \text{ and } \|x - y\|_\infty \leq \delta \text{ imply } \|f(x) - f(y)\|_\infty \leq \epsilon.$$

Theorem 4.1. Suppose that the algorithm works on the VK'-triangulation of $S^n \times R_+^m$ with meshsize smaller than the above δ. Then, under Assumption (LRC), it terminates within a finite number of steps

Proof. It is sufficient to show that there does not exist any s-complete simplex in $A(s) \cap D(T)$. Suppose to the contrary that there is an s-complete simplex $\sigma(a, \pi)$ with vertices $w^1, w^2, \ldots, w^{t+1}$ in $D(T)$. This implies that $s_1 \leq 0$ and $s_{2k} = +1$ for some $k \in I_m$. According to Eq. (6), we have that the piecewise linear approximation F_1 at $w = \sum_{i=1}^{t+1} \lambda_i w^i \in \sigma$ is nonpositive, i.e., $F_1(w) \leq 0$. Because $w \in D(T)$, there exists some j for which $f_{1j}(w) \geq \varphi$ according to Lemma 4.1. It follows that

$$f_{1j}(w^i) \geq \varphi/2, \qquad \text{for all vertices } w^i \text{ of } \sigma.$$

Since

$$F_1(w)=\sum_{i=1}^{t+1} \lambda_i f_1(w^i),$$

and

$$\sum_{i=1}^{t+1} \lambda_i=1, \qquad \lambda_i \geq 0, \text{ for } i=1,\ldots,t+1,$$

we obtain

$$F_{1j}(w) \geq \varphi/2>0.$$

This is a contradiction. □

It is easily seen from Theorem 4.1 that as the meshsize of the VK'-triangulation of $S^n \times R^m_+$ goes to zero, the endpoints of the paths $\bar{P}$ followed by the algorithm yield a subsequence that converges to an equilibrium.

5. Applications

In this section, we apply the adjustment process to economies with constant returns to scale. It may be worth mentioning that this process can also be used to find an equilibrium in an economy with increasing returns to scale and converges for any starting point under the condition stated in Hofkes (Ref. 8). Let us consider an economy with a finite number of consumers, m firms indexed by $i=1,\ldots,m$, each having constant returns to scale production functions, and $n+1$ commodities indexed by $j=1,\ldots,n+1$. Consumers are assumed to be endowed with the commodities. Given a price vector $p\in R^{n+1}_+\backslash\{0\}$ with p_j denoting the price of commodity j, let $d(p)$ denote the total demand of the consumers, where $d_j(p)$ is the demand for commodity $j\in I_{n+1}$, and let $z(p)$ be the total demand $d(p)$ minus the total initial endowments. Standard assumptions on z are as follows:

(A1) z is continuous in $p\in R^{n+1}_+\backslash\{0\}$;
(A2) z is homogeneous of degree zero; i.e., $z(\lambda p)=z(p)$ for any $\lambda>0$ and $p\in R^{n+1}_+\backslash\{0\}$;
(A3) $p'z(p)=0$, for every $p\in R^{n+1}_+\backslash\{0\}$; this is the Walras law.

Commodities in the economy can be produced by the firms. A production activity of firm i, $i\in I_m$, at price vector $p\in R^{n+1}_+\backslash\{0\}$, is characterized by an input–output $(n+1)$-vector $a^i(p)$ whose negative components correspond to the amounts of inputs and whose positive components to the amounts of outputs per unit production. Then, $p'a^i(p)$ represents the profit of firm i, $i \in I_m$, per unit production. Moreover, a^i, $i \in I_m$, is homogeneous in p of degree zero, concave and continuous on $R_+^{n+1}\backslash\{0\}$. Let y be a nonnegative m-vector of production levels, and let $A(p)$ be the $(n+1) \times m$ matrix $[a^1(p), \ldots, a^m(p)]$. Hence, $A(p)y$ denotes the net supply of the production side at price vector p and production level vector y. For this economy, we call a price vector p^* and production level vector y^* an equilibrium if, for each commodity, demand is at most equal to endowment plus net supply of the production side and no production activity makes a positive profit. Let the net excess demand function $f_1: R_+^{n+1}\backslash\{0\} \times R_+^m \to R^{n+1}$ be defined by

$$f_1(p, y) = z(p) - A(p)y,$$

i.e., $f_1(p, y)$ is the excess demand of the consumption side at p minus the net supply of the production side at (p, y). Further, let the profit function $f_2: R_+^{n+1}\backslash\{0\} \times R_+^m \to R^m$ be defined by

$$f_2(p, y) = A^t(p)p,$$

i.e., $f_2(p, y)$ is the vector of profits at p per unit activity. For a detailed description of the model, we refer to Van der Laan and Kremers (Ref. 11).

Definition 5.1. A pair $(p^*, y^*) \in R_+^{n+1}\backslash\{0\} \times R_+^m$ is an equilibrium if

(i) $f_1(p^*, y^*) \leq 0$,
(ii) $f_2(p^*, y^*) \leq 0$.

Because of the homogeneity of degree zero of z and a^i, $i = 1, \ldots, m$, we have that, if (p^*, y^*) is an equilibrium, then also $(\lambda p^*, y^*)$ is an equilibrium pair for any $\lambda > 0$. So this permits us to normalize the price vectors to the n-dimensional unit simplex S^n. Now, the problem is reduced to the one that we discussed in Section 2. For the existence of an equilibrium in this economy, the following no free production assumption is introduced in Ref. 11.

Assumption (F). No Production without Input. For any $p \in S^n$, $A(p)y \geq 0$ and $y \geq 0$ implies that $y = 0$.

Taking $(u, v) \in S^n \times R_+^m$ and a positive vector $b \in R_+^m$ as described in Section 2, we will show that, under Assumption (F), there exists an equilibrium in the economy with constant returns to scale via the adjustment process (3). To do so, it suffices to prove the following assertion.

Theorem 5.1. The path P in $S^n \times R_+^m$ is bounded.

Proof. Suppose that the path P is unbounded. Then, without loss of generality, there is some sign vector $s \in S$ with $s_1 \leq 0$ such that $A(s)$ contains a sequence $\{(p^k, y^k)\}_1^\infty$, with some of the components of y^k going to infinity. Since S^n is compact, the sequence p^k has a subsequence converging to a cluster point q in S^n. Because $(p^k, y^k) \in A(s)$, we have that

$$f_{1j}(p^k, y^k) \leq 0, \qquad \text{for all } j.$$

Hence, there exist nonnegative numbers μ_j^k for all $j \notin I^0(s_1)$ such that

$$f_1(p^k, y^k) - \sum_{j \notin I^0(s_1)} \mu_j^k s_{1j} e_1(j)$$
$$= z(p^k) - A(p^k)y^k + \sum_{j \in I^-(s_1)} \mu_j^k e_1(j) = 0. \tag{7}$$

Since p^k has a subsequence converging to q and z is continuous, the system (7) can only have a solution for all k if the homogeneous system of linear equations

$$-A(q)y + \sum_{j \in I^-(s_1)} \mu_j e_1(j) = 0 \tag{8}$$

has a nonzero solution $y_i^* \geq 0$, $i \in I_m$, and $\mu_j^* \geq 0, j \in I^-(s_1)$. On the one hand, if $y^* = 0$, there exists at least one component of μ_j^* greater than zero, contradicting the system (8). On the other hand, if $y^* \neq 0$, it easily follows from Assumption (F) that at least one component of $A(q)y^*$ is less than zero, which is also in contradiction with the system (8). From these results, the system (7) does not have a nonzero nonnegative solution. This completes the proof. □

Theorem 5.1 indicates that the path P is bounded and therefore leads to another endpoint which must be an equilibrium. Of course, the adjustment process can also be applied to the special case of linear production technologies. In that case, the matrix $A(p)$ is independent of p. Then, the process converges for any starting point under the standard assumption that there can be no production without input; see, e.g., Refs. 4, 7, 16.

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